On The Analysis and Synthesis of Single-Element-Kind Networks*

E. A. GUILLEMIN†

INTRODUCTION

THE following discussion pertains to networks containing only one kind of element—R, L, or C. To be specific we shall consider the element kind to be resistive and regard the branch parameters to be given in terms of their conductance values since the node basis will be chosen to characterize network equilibrium. No loss in generality is thereby implied, it being understood that familiar methods of source transformation and the duality principle are available for generalizing end results in ways appropriate to the effective handling of any given physical situation.

DETERMINATION OF THE NODE CONDUCTANCE MATRIX FROM BRANCH CONDUCTANCES OR VICE VERSA

The first topic that we shall consider deals with establishing simple ways of relating topology and branch conductance values to the node conductance matrix and vice versa. The node conductance matrix is the parameter matrix characterizing the equilibrium equations in terms of a chosen set of node-pair voltages. The number of these may be equal to or less than the number of branches in a tree appropriate to the given network graph, since one need not consider all of the independent node pairs to be points of access. If we do consider all of them to be accessible node pairs then a node conductance matrix of order n pertains to a network having a total of $n + 1$ nodes (for which the number of branches in any tree equals n). We shall begin by considering this special situation since familiarity with it provides an essential background for the more general problem.

If the total number of branches is b and the cut-set matrix having n rows and b columns is denoted by α, then the node conductance matrix is given by the familiar expression

$$G = \alpha g_b \alpha_t \qquad (1)$$

in which g_b is the diagonal branch conductance matrix of order b, and α_t is the transpose of α. Since α can be chosen in a large number of ways for the same given network, characterization of the precise form and properties of G is not simple. However, for a fixed geometrical tree configuration, α is essentially fixed in form except for a rearrangement of rows (the arrangement of columns being immaterial since it affects only the identi-

ties of the diagonal elements in g_b). Hence for a fixed tree geometry, G can vary only in a rearrangement of its rows and columns, and a change in the reference direction for a tree-branch (node-pair) voltage merely causes all element values in a corresponding row and column to change. We will regard such changes in G as not altering its fundamental form and hence consider the number of distinct fundamental forms of G as being equal to the number of distinct geometrical tree configurations constructible for a given n (the number of distinct patterns constructible with n match sticks.)

Out of the variety of possible tree configurations, two particular ones are of primary interest: the "starlike" tree in which all branches have the same node in common, and the "linear" tree in which successive branches have one node in common. The first of these implies a node-to-datum set of node-pair voltages and gives rise to the so-called "dominant" matrix G in which all nondiagonal terms are negative, the diagonal ones positive, and sums of elements in any row or column are non-negative. As mentioned above, sign changes resulting from the multiplication of corresponding rows and columns by -1 are disregarded since they can easily be recognized and corrected.

Fig. 1 shows a graph for $n = 5$ in which the branches of the starlike tree are numbered 11, 22, 33, 44, 55. This is a so-called "full" graph, in that branches connect each node with all other nodes, the total number of branches being $n(n + 1)/2$, the same as the total number of distinct elements in the symmetrical matrix G of order n. Branches connecting nodes i and k are labeled ik; and we shall denote conductance values of the branches by g_{ik} corresponding to the branch numbering given in this graph.

In Fig. 2 this same graph is redrawn with the nodes arranged upon a straight line so that comparison with the branch numbering for the choice of a linear tree becomes easier. The latter is shown in Fig. 3.

In the node-to-datum arrangement with the starlike tree, the algebraic relations between the branch conductances g'_{ik} and the elements G'_{ik} of the dominant matric G' are obvious, and are well known to be given by

$$g'_{ik} = -G'_{ik} \quad \text{for} \quad i \neq k \qquad (2)$$

and

$$g'_{ii} = \sum_{k=1}^{n} G'_{ik} \quad \text{or} \quad G'_{ii} = \sum_{k=1}^{n} g'_{ik} \qquad (3)$$

where primes on these quantities are used because we want to reserve the same notation without a prime for the corresponding quantities appropriate to a linear tree.
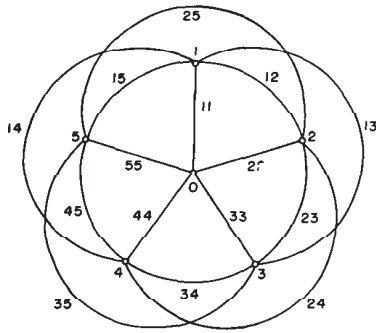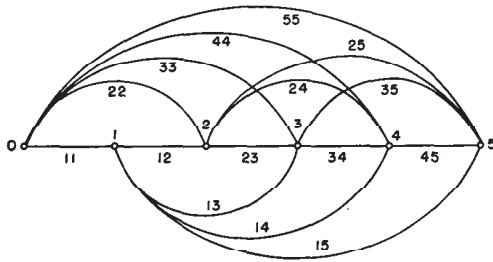
Fig. 1.

Fig. 2.

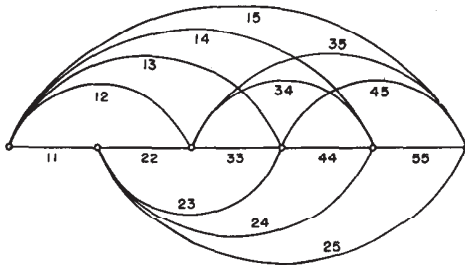
Fig. 3.

Denoting the node-to-datum voltages by a matrix e', the pertinent equilibrium equations read

$$G'e' = i'_s \tag{4}$$

in which the elements of i'_s are current sources across tree branches. For the linear tree of Fig. 3 the corresponding equilibrium equations are written

$$Ge = i_s, \tag{5}$$

and if we denote the transformation from one set of node-pair voltages to the other by the matrix equation

$$e = Te', \tag{6}$$

then inspection of Figs. 2 and 3 shows that the transformation matrix T has the form

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & & -1 & 1 \end{bmatrix} \tag{7}$$

with the inverse

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{8}$$

From (5) and (6) we get

$$T_t G T e' = T_t i_s = i'_s, \tag{9}$$

(the identification of $T_t i_s$ with i'_s is familiarly required by the condition of power invariance) and thus we have by comparison with (4)

$$G' = T_t G T \tag{10}$$

in which T_t is the transpose of T.

Through use of (7) and (8) we can now find

$$TGT = TT_t^{-1}G'$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & -1 & 0 \end{bmatrix} \times G' \tag{11}$$

whereupon the relations (2) and (3) yield

$$TGT = \tag{12}$$

n which the cross-hatched region below the principal diagonal contains elements in which we have no interest. From a comparison of Figs. 2 and 3, noting particularly the differences in notation for the branch conductances, we see that the result just obtained may be written

$$TGT = \tag{13}$$

which represents relationships between branch conductances in the graph of Fig. 3 and elements in the node conductance matrix appropriate to a linear tree, that are comparable in simplicity with the familiar ones pertinent to a conductance matrix based upon a node-to-datum set of voltage variables. In fact, if we tilt the matrix 13 so that its principal diagonal becomes horizontal, and redraw
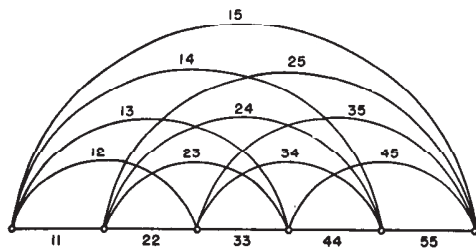
Fig. 4.

the graph of Fig. 3 in the pyramidal form shown in Fig.4, then the identification of elements in this matrix with pertinent branches in the associated network[1] becomes strikingly evident, as does also the fact that the number of distinct elements in a conductance matrix G of order n exactly equals the number of branches in a full graph with n tree branches.

Recognition of the latter relationship, incidentally, makes it clear that we can always obtain branch conductance values appropriate to a given cut-set matrix α and node conductance matrix G, by writing matrix equation (1) in the equivalent algebraic form

$$G_{ik} = \sum_{\nu=1}^{b} \alpha_{i\nu}\alpha_{k\nu}g_{\nu} \tag{14}$$

(in which g_{ν}, the branch conductances, are elements in the diagonal matrix g_b) and solving this set of $n(n + 1)/2$ simultaneous equations for the same number of unknown g_{ν}'s. In the case where α is based upon a linear tree, one arrives in this way (after appropriate manipulation) at the same result made evident in (13) and Fig. 4.

This result affords an equally simple numerical procedure for computing G_{ik}'s or vice versa since the operation on rows and columns demanded by the transformation matrix T or T^{-1} is so easy to carry out. Suppose we denote the matrix 13 by g and assume as an example

$$g = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ & 1 & 2 & 2 & 1 \\ & & 1 & 2 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}. \tag{15}$$

Then by inspection

$$T^{-1}g = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ & 3 & 5 & 4 & 2 \\ & & 6 & 6 & 3 \\ & & & 7 & 4 \\ & & & & 5 \end{bmatrix} \tag{16}$$

[1] This result can also be obtained from the well-known physical interpretation of what the elements in G represent.

and so we have the conductance matrix

$$G = \begin{bmatrix} 9 & 8 & 6 & 3 & 1 \\ & 14 & 11 & 6 & 2 \\ & & 15 & 9 & 3 \\ & & & 11 & 4 \\ & & & & 5 \end{bmatrix}. \tag{17}$$

In (15) and (16) the elements below the principal diagonal are of no interest and are not involved in the manipulations. In (17) they are of interest, of course, and can readily be inserted knowing that G is symmetrical.

Proceeding in the opposite direction we obtain from (17)

$$TG = \begin{bmatrix} 9 & 8 & 6 & 3 & 1 \\ & 6 & 5 & 3 & 1 \\ & & 4 & 3 & 1 \\ & & & 2 & 1 \\ & & & & 1 \end{bmatrix} \tag{18}$$

and so

$$TGT = g = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ & 1 & 2 & 2 & 1 \\ & & 1 & 2 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \tag{19}$$

which is what we started out with.

In going from (15) to (16) we write down the first row in (15), then add to this one the second row in (15) to form the second row in (16), then add to this one the third row in (15) to form the third in (16), and so forth. Having formed (16), we construct (17) by columns, starting by writing down the fifth column in (16), then add to this one the fourth column in (16) to form the fourth in (17), and so forth as was done with the rows.

In the opposite direction, starting with (17), its first row is the same in (18). The second row in (18) is the second minus the first in (17) (ignoring the absent term); the third row in (18) is the third minus the second in (17); the fourth is the fourth minus the third in (17), and so forth. Analogous operations on columns, starting with the fifth and working towards the first, yield the transformation from (18) to (19). With a little practice, these operations become very simple to carry out. Only addition and subtraction are involved.

The process is about as easy to perform, and the conditions leading to positive elements in g are almost as easily recognizable as are those pertaining to a dominant G matrix appropriate to a graph with starlike tree. These conditions on a G matrix appropriate to a linear tree will be implied by designating G to be a "uniformly tapered" matrix, the reason for the choice of this term being evident in the numerical example just given.

Response Functions Directly Related to Branch Conductance

In an analysis problem, having formed the matrix G for a given graph and its branch conductance values, we are next interested in evaluating elements in the inverse of G — the open-circuit driving-point and transfer impedances for the chosen terminal pairs. The following manipulations are aimed at devising a computational scheme whereby these impedances may be obtained from the branch conductance values with a minimum number of additions, multiplications, and divisions.

We begin by writing for the symmetrical matrix G the representation

$$G = A \times A_t \tag{20}$$

$$T \times A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ (a_{21} - a_{11}) & a_{22} & 0 & \cdots & 0 \\ (a_{31} - a_{21}) & (a_{32} - a_{22}) & a_{33} & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (a_{n1} - a_{n-1,1}) & (a_{n2} - a_{n-1,2}) & (a_{n3} - a_{n-1,3}) & \cdots & a_{nn} \end{bmatrix} \tag{23}$$

$$A_t \times T = \begin{bmatrix} (a_{11} - a_{21}) & (a_{21} - a_{31}) & (a_{31} - a_{41}) & \cdots & (a_{n-1,1} - a_{n1}) & a_{n1} \\ -a_{22} & (a_{22} - a_{32}) & (a_{32} - a_{42}) & \cdots & (a_{n-1,2} - a_{n2}) & a_{n2} \\ 0 & -a_{33} & (a_{33} - a_{34}) & \cdots & (a_{n-1,3} - a_{n3}) & a_{n3} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & 0 & \cdots & (a_{n-1,n} - a_{nn}) & a_{nn} \end{bmatrix}. \tag{24}$$

and assume for A the triangular form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}. \tag{21}$$

Since a triangular matrix is easy to invert, the inverse of G, which is given by

$$G^{-1} = A_t^{-1} \times A^{-1}, \tag{22}$$

is likewise more easily obtainable from the representation (20) inasmuch as the elements in A are readily computed noting that G is the Grammian matrix formed from the rows of A. Thus the elements in A are found by a simple recursion process, and so are those in the inverse of A which again is triangular and has the same form as A.

Instead of pursuing the details of this process, however (the reader can do this for himself) we wish to show how we can tie the branch conductance values into this representation so that, in an analysis problem, we can omit the formation of the matrix G altogether and proceed directly with the formation of G^{-1}. Thus, with (13) and (20) in mind, and the specific form of T given by (7) before us, we construct the products

The second of these matrices is almost the negative transpose of the first. In fact if we add to the first matrix a last row with the elements $-a_{n1}$, $-a_{n2}$, $\cdots$, $-a_{nn}$, and then ignore the first row, its negative transpose is the second matrix. This fact suggests that we consider the matrix (23) with the stated additional row, namely the matrix with $n + 1$ rows and n columns given by

$$H = \begin{bmatrix} a_{11} & 0 & 0 & \cdot & 0 \\ (a_{21} - a_{11}) & a_{22} & 0 & \cdot & 0 \\ (a_{31} - a_{21}) & (a_{32} - a_{22}) & a_{33} & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (a_{n1} - a_{n-1,1}) & (a_{n2} - a_{n-1,2}) & (a_{n3} - a_{n-1,3}) & \cdot & a_{nn} \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdot & -a_{nn} \end{bmatrix}. \tag{25}$$

In this matrix all columns add to zero; if the vector set defined by rows is denoted by h_0, h_1, h_2, $\cdots$ h_n, then these vectors form the sides of a closed polygon in n-dimensional space. The branch conductance values in the matrix (13)

are seen to be given by the scalar products

$$g_{ik} = -h_{i-1} \cdot h_k \quad \text{for} \quad i \leq k \leq n$$
$$\text{and} \quad i = 1, 2, \cdots n. \quad (26)$$

More specifically, if we designate the elements in H as indicated in

$$H = \begin{bmatrix} h_{01} & 0 & 0 & \cdot & 0 \\ h_{11} & h_{12} & 0 & \cdot & 0 \\ h_{21} & h_{22} & h_{23} & \cdot & \vdots \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & h_{n-1,n} \\ h_{n1} & h_{n2} & h_{n3} & \cdot & h_{nn} \end{bmatrix}, \quad (27)$$

we obtain for the g_{ik} the expressions

$$g_{ik} = -\sum_{\nu=1}^{i} h_{i-1,\nu} h_{k\nu} \quad \text{for} \quad k \geq i. \quad (28)$$

Since we are interested in computing elements in the matrix H from g_{ik} values in a given network (because the elements of A and of its inverse are then readily obtainable) we manipulate (28) as follows. First we split off the last term in the sum and have

$$g_{ik} = -\sum_{\nu=1}^{i-1} h_{i-1,\nu} h_{k\nu} - h_{i-1,i} h_{ki} \quad (29)$$

or

$$-h_{i-1,i} h_{ki} = g_{ik} + \sum_{\nu=1}^{i-1} h_{i-1,\nu} h_{k\nu}. \quad (30)$$

For a particular column of H (fixed value of i) it will be noticed that the right-hand side of this equation involves only coefficients h in preceding columns (up to and including column $i - 1$). Eq. (30) would thus be suited for the sequential computation of the h coefficients if it were not for the factor $h_{i-1,i}$ on the left. This awkwardness may be removed by introducing quantities

$$p_{ki} = -h_{i-1,i} h_{ki} = g_{ik} + \sum_{\nu=1}^{i-1} h_{i-1,\nu} h_{k\nu}. \quad (31)$$

Since the columns of H add to zero, we have

$$\sum_{k=i-1}^{n} h_{ki} = 0, \quad \text{or} \quad h_{i-1,i} = -\sum_{k=i}^{n} h_{ki} \quad (32)$$

and thus

$$\sum_{k=1}^{n} p_{ki} = -h_{i-1,i} \sum_{k=i}^{n} h_{ki} - h_{i-1,i}^2. \quad (33)$$

From (31),

$$h_{ki} = \frac{p_{ki}}{-h_{i-1,i}}. \quad (34)$$

Using this relation and (33) to form

$$h_{i-1,\nu} h_{k\nu} = \frac{p_{i-1,\nu} p_{k\nu}}{h_{\nu-1,\nu}^2} = \frac{p_{i-1,\nu} p_{k\nu}}{\sum\limits_{k=\nu}^{n} p_{k\nu}}, \quad (35)$$

substitution back into (31) yields the result

$$p_{ki} = g_{ik} + \sum_{\nu=1}^{i-1} \left[\frac{p_{i-1,\nu} p_{k\nu}}{\sum\limits_{k=\nu}^{n} p_{k\nu}} \right]; \quad (k \geq i = 1, 2, \cdots n). \quad (36)$$

Note that for $i = 1$ the sum drops out and we have simply $p_{k1} = g_{1k}$. The matrix with coefficients p_{ki} having the form

$$P = \begin{bmatrix} p_{11} & 0 & 0 & \cdot & 0 \\ p_{21} & p_{22} & 0 & \cdot & 0 \\ p_{31} & p_{32} & p_{33} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \circ \\ p_{n1} & p_{n2} & p_{n3} & \cdot & p_{nn} \end{bmatrix} \quad (37)$$

may be calculated sequentially starting with the elements in the first column, then those in the second, and so forth since the formula (36) for any fixed value of the index i involves only elements in the columns 1 to $i - 1$. The sum in the denominator of the summand in (36) is simply the sum of all elements in the νth column. Hence in the computational procedure, each time the elements in an additional column are calculated, their sum may also be recorded below it, so that its value is readily available for computation of the next column.

The elements of the matrix A, (21), may now be expressed directly in terms of the p_{ki}. From the form of the matrix H, (25), and the notation in (27) we have first of all

$$a_{ik} = \sum_{\nu=k-1}^{i-1} h_{\nu k} = h_{k-1,k} + \sum_{\nu=k}^{i-1} h_{\nu k}$$

$$= \sum_{\nu=k}^{i-1} h_{\nu k} - \sum_{\nu=k}^{n} h_{\nu k} = -\sum_{\nu=i}^{n} h_{\nu k} \quad (38)$$

in which the relation in (32) is made use of. Substituting for $h_{\nu k}$ from (34) we have, with an assist from (33),

$$a_{ik} = \frac{\sum\limits_{\nu=i}^{n} p_{\nu k}}{h_{k-1,k}} = \frac{\sum\limits_{\nu=i}^{n} p_{\nu k}}{\sqrt{\sum\limits_{n=k}^{n} p_{\nu k}}} \quad \text{for} \quad i \geq k = 1, 2, \cdots n. \quad (39)$$

Appearance of the radical in this expression does not contradict the well-known requirement that response in a lumped network be a rational function of the branch conductances since the impedances which we shall presently compute are quadratic functions of the a_{ik}, as is also evident from (22). For this reason it is advisable not to compute the a_{ik}'s until an evaluation of the desired response function in terms of these coefficients is made and the radicals are eliminated.

If the inverse of the matrix A is denoted by

$$A^{-1} = B = \begin{bmatrix} b_{11} & 0 & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & b_{n3} & & \cdots & b_{nn} \end{bmatrix}, \quad (40)$$

we have the following relations between the a_{ik} and b_{ik}

$$a_{11}b_{11} = 1 \quad (41)$$

$$\left. \begin{array}{l} a_{11}b_{21} + a_{21}b_{22} = 0 \\ a_{22}b_{22} = 1 \end{array} \right\} \quad (42)$$

$$\left. \begin{array}{l} a_{11}b_{31} + a_{21}b_{32} + a_{31}b_{33} = 0 \\ a_{22}b_{32} + a_{32}b_{33} = 0 \\ a_{33}b_{33} = 1 \end{array} \right\}, \quad (43)$$

and so forth. Or, starting at the opposite end,

$$a_{nn}b_{nn} = 1 \quad (44)$$

$$\left. \begin{array}{l} a_{n-1,n-1}b_{n,n-1} + a_{n,n-1}b_{nn} = 0 \\ a_{n-1,n-1}b_{n-1,n-1} = 1 \end{array} \right\} \quad (45)$$

$$\left. \begin{array}{l} a_{n-2,n-2}b_{n,n-2} + a_{n-1,n-2}b_{n,n-1} + a_{n,n-2}b_{nn} = 0 \\ a_{n-2,n-2}b_{n-1,n-2} + a_{n-1,n-2}b_{n-1,n-1} = 0 \\ a_{n-2,n-2}b_{n-2,n-2} = 1 \end{array} \right\}, \quad (46)$$

and so forth.

The open-circuit resistance matrix, which is the inverse of G, is given by

$$R = G^{-1} = B_t \times B. \quad (47)$$

The elements of R in which we are particularly interested are

$$r_{nn} = b_{nn}^2 \quad (48)$$

$$r_{n-1,n} = b_{n,n-1}b_{nn} \quad (49)$$

$$r_{n-2,n} = b_{n,n-2}b_{nn}. \quad (50)$$

The first of these is a driving-point function; the other two are transfer functions. Because of the implied linear tree upon which the terminal pairs are based, we see more particularly that (49) is the open-circuit transfer impedance of a grounded two terminal-pair (three-terminal network) while (50) is the open-circuit transfer impedance of an arbitrary two terminal-pair network since the two terminal pairs involved are not adjacent and hence do not have a terminal in common.

From (44), (48), and (39) we have for the driving-point impedance (which incidentally can, through appropriate branch numbering, be the impedance across any chosen branch in the given network) the surprisingly simple result

$$r_{nn} = \frac{1}{a_{nn}^2} = \frac{1}{p_{nn}}. \quad (51)$$

For the transfer function (49) we find straightforwardly

$$\frac{r_{n-1,n}}{r_{nn}} = -\frac{a_{n,n-1}}{a_{n-1,n-1}} = \frac{-p_{n,n-1}}{(p_{n-1,n-1} + p_{n,n-1})} \quad (52)$$

and for the one given by (50) we get

$$\frac{r_{n-2,n}}{r_{nn}} = \frac{a_{n,n-1}a_{n-1,n-2} - a_{n,n-2}a_{n-1,n-1}}{a_{n-1,n-1}a_{n-2,n-2}}$$

$$= \frac{p_{n,n-1}p_{n-1,n-2} - p_{n,n-2}p_{n-1,n-1}}{(p_{n-1,n-1} + p_{n,n-1})(p_{n-2,n-2} + p_{n-1,n-2} + p_{n,n-2})}. \quad (53)$$

Regarding numerical computation of these quantities, the determination of all elements in the matrix P, (37), by the formula (36) is found to involve

$$(n - 1) \cdot 1 + (n - 2) \cdot 2 + (n - 3) \cdot 3 + \cdots + 1 \cdot (n - 1)$$

$$= \sum_{x=1,2\ldots}^{n-1} \frac{x(x + 1)}{2} = \frac{n(n - 1)(n + 1)}{6} \quad (54)$$

multiplications, the same number of divisions, and

$$\frac{n(n - 1)(n + 1)}{6} + \frac{n(n - 1)}{2} = \frac{n(n - 1)(n + 4)}{6} \quad (55)$$

additions or a total of

$$\frac{n(n - 1)(n + 2)}{2} \quad (56)$$

operations altogether. The number of additional operations involved in the computation of driving-point or transfer functions (48), (49), or (50) is small in any case and is evident from (51)–(53) in which it should be noted that the sums appearing in the denominators of the last two of these are already available and do not require further addition. Thus computation of r_{nn} requires one additional division, $r_{n-1,n}$ requires additionally one multiplication and one division, and $r_{n-2,n}$ requires additionally four multiplications, a subtraction, and one division.

This method for the calculation of network response thus appears to be computationally more economical than any other known method, especially the recently revived and much talked about Kirchhoff combinatorial method and trivial variations thereof. The common denominator in the expressions obtained by the latter method have as many terms as the graph has enumerable trees (each term being the product of n branches). Thus in a full graph with $n = 3$ there are 16 trees and this denominator alone involves 15 additions plus 32 multiplications while our formula (56) yields 15 for all operations (additions, multiplications, and divisions).

As a simple example, let us take the graph of Fig. 5 for which $n = 3$. Numbers on the branches are conductance values in mhos and the branch numbering is understood to follow the pattern set in Fig. 4. Let the problem be to find the input impedance across the 1/2-mho tree-branch number 3 (which by rearrangement could be any other branch as well) and the open-circuit transfer impedances between this branch and the other two tree branches (which can also be any other two branches). Construction of the P matrix according to (36) takes the form indicated

in the schedule below

$$
\begin{array}{ll}
1 & \\
2 & \frac{2}{3} + \frac{1}{3} = 1 \\
3 & \frac{1}{2} + \frac{1}{2} = 1 \quad \frac{1}{2} + 1 + \frac{1}{2} = 2 \\
6 & 2
\end{array}
\qquad (57)
$$

from which (51)–(53) yield

$$r_{33} = 1/2 \quad \text{ohm}$$

$$2r_{23} = -1/2 \quad \text{ohm}$$

$$2r_{13} = \frac{2-3}{2 \cdot 6} = -1/12 \quad \text{ohm.}$$

The minus signs in the last two results arise from the implied reference directions for the tree branches as shown in Fig. 5.
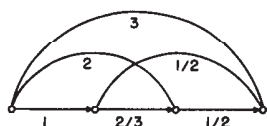

Fig. 5.

REALIZATION OF AN nTH ORDER MATRIX G BY AN $n + 1$-NODE NETWORK

If a given node conductance matrix G is known to be appropriate to a star-like tree, then the necessary and sufficient realizability conditions (as an $n + 1$-node network) are simply that it be a dominant matrix; if it is known to be based upon a linear tree, then the uniformly tapered property described above is necessary and sufficient for its realization. For any given order n there are a finite number of distinct geometrical tree configurations; the given G matrix must be based upon one of these and satisfy pertinent realizability conditions if a corresponding network with all positive elements is to exist.

If we know the geometrical tree configuration upon which the matrix G is based, then we can readily write the appropriate transformation matrix T connecting the node-pair voltages in that tree to those in a starlike or a linear tree and, by means of a congruent transformation like that expressed in (10), transform G to a form appropriate to either of these basic tree configurations, whereupon the question regarding its realizability is readily answered, and, if answered in the affirmative, the corresponding network is constructable straightforwardly.

The problem of testing and realizing a given G matrix by an $n + 1$-node network, therefore, will be solved if we can devise a method for discovering the geometrical tree configuration upon which that G matrix is based. This we shall now proceed to do, and we shall see, incidentally, that the pertinent tree configuration is unique if G is nondegenerate in the sense that all of its elements are nonzero. Thus our method, as we shall see, is based upon the recognition that there exists a one-to-one correspondence (properly interpreted, of course) between the pertinent tree configuration and the algebraic sign distribution among elements in the G matrix. This algebraic sign pattern enables us to recognize the geometrical configuration of the tree upon which G is based, and G must be realizable with that tree configuration if it is realizable at all.

Obviously, if G contains zeros, there is an ambiguity in the algebraic sign pattern, and the possibility exists that more than one tree may be appropriate. Although one can still apply the method to obtain systematically all appropriate trees in a situation of this sort, the process loses its compactness and we shall, therefore, assume for the time being that G has no zero elements.

The easiest way to see that a definite algebraic sign pattern among the elements in G is linked with a given geometrical tree configuration, is through physical rather than analytical reasoning. Since G is a *short-circuit* driving-point and transfer matrix, we visualize all n terminal pairs across tree branches provided with short-circuiting links and remind ourselves that the values of currents in these links, per volt of ideal voltage source in one of them, equal numerically the elements in G inclusive of their algebraic signs relative to chosen reference arrows on the tree branches.

Thus, with the unit voltage source in the short-circuiting link across tree branch no. 1, the currents in the various links numerically equal the elements in row 1 of G. With the unit voltage source in the link across tree branch no. 2, the currents in the n short-circuiting links in value equal the elements in row 2 of G, etc. The voltage rise of the source is made to coincide in direction with the reference arrow on the tree branch in parallel with it, and any resulting current is positive if its direction in the pertinent short-circuiting link agrees with the reference arrow on the branch alongside of it, and it is a negative current if its direction is opposite to this reference arrow.
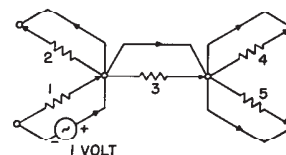

Fig. 6.

As an example, Fig. 6 shows a tree for $n = 5$ with short-circuiting links across its branches and a voltage source applied to terminal pair no. 1. If we visualize the presence of the rest of the branches associated with this tree in a full graph, it is clear by inspection that the various short-circuit currents have directions as indicated by arrows in the short-circuiting links; and thus we see that all elements in the first row of a G matrix appropriate to this tree (with the assumed reference arrows) are positive.

If we shift the voltage source into the short-circuiting link across branch 2 with its polarity in the same relation to the reference arrow on that branch as is shown for branch no. 1 in Fig. 6, then it is equally simple to see by inspection that the resulting currents for branches 1 and 2 are in positive directions while those in all the other short-circuiting links are negative. In the second row of G, therefore, the first two elements are positive and the rest are negative. Thus it is a simple matter to establish all algebraic signs for the elements of G; and one becomes convinced, incidentally, that this sign pattern has nothing to do with element values but is uniquely fixed by the geometrical tree configuration except for interchanging rows and columns (renumbering of tree branches), and the multiplication of rows and columns by minus signs (changing reference arrows on the tree branches), which, as we shall see presently, are trivial operations so far as recognition of the pertinent tree configuration is concerned.

In order to facilitate the correlation of algebraic sign patterns with geometrical tree configurations, we observe that we can dispense with drawing the tree branches in sketches like the one in Fig. 6. It is sufficient to draw lines for branches as we are accustomed to do in network graphs and to regard these as the short-circuiting links, their reference arrows being included in the usual manner. A voltage ssurce as well as all other branches in the full graph can easily be imagined to be in their proper places and the directions of pertinent currents can with a little practice be deduced by inspection.
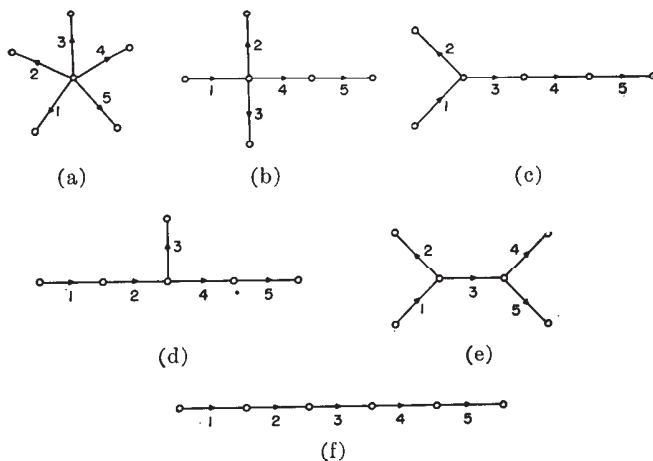

Fig. 7.

In Fig. 7 are drawn all six tree configurations for $n = 5$. We can imagine the branches as being water pipes. If in the starlike tree [Fig. 7(a)] we squirt water into pipe no. 1 in its reference direction it will obviously flow in the positive reference directions in all of the four other pipes. Hence in the first row of G all elements are plus. If we squirt water in the reference direction through pipe no. 2, it flows positively through pipe no. 1 and negatively through the other three. Thus in the second row of G the first two elements are plus and the rest are minus. This

homely analogy makes the process of establishing sign patterns for all trees fast and effortless. The resulting sign patterns thus obtained are indicated as follows by what we might call "sign matrices":

$$(59)$$

$$(60)$$

Since these matrices are symmetrical it is necessary only to record signs above the principal diagonal.

Signs on the principal diagonal are always plus for obvious reasons. Observe further that we have chosen reference arrows for the tree branches in such a way that the first row is always a row of plus signs. In an arbitrarily given G matrix this state of affairs need not be fulfilled, but we can always multiply rows (and corresponding columns) by minus signs to fulfill this condition and thus convert the given matrix to a sort of normal or basic form with regard to its algebraic sign pattern. This step eliminates once and for all those trivial variants of the given matrix G which might stem from sign changes of its elements in rows (and in corresponding columns). We may regard the process of making all signs in the first row positive, as one of reducing the reference arrows on the branches of the implied tree to a common basic pattern.

Except for row and column interchanges (renumbering of tree branches) each sign matrix uniquely specifies a geometrical tree pattern and vice versa; and we shall presently describe a simple way of constructing the tree from a given sign matrix. In the meantime it is interesting to observe that *the linear tree is the only one for which all signs in the G matrix are positive.* If a given G matrix has all positive elements (when its sign pattern is normalized as just described) then it must be realizable with a linear tree if it is realizable at all; that is to say, it must be a uniformly tapered matrix or else it has no realization in an $n + 1$-node network.

In the given G matrix with all-positive elements the arrangement of rows and columns may not be such as to give G the uniformly tapered form even though it is basically appropriate to a linear tree, since the branch numbering in that tree may not be consecutive. However, one can readily perceive the needed rearrangement of elements in the first row of G that yields a uniform taper from left to right and thus the column and row interchanges in G that are needed to put it into uniformly tapered form are evident. This revised form of G must then yield positive branch conductance values according to the transformation (13) [with T given by (7)] if it is realizable at all.

HOW TO GROW A TREE FROM A GIVEN SIGN MATRIX

The procedure for constructing a tree from a given sign matrix is similar in some respects to a method already developed for determining the graph pertinent to a given cut-set matrix.[2] There, however, the growth pattern for the tree must first be established while in the present situation one can proceed at once with the growth process for the pertinent tree since any given sign matrix, in contrast with a cut-set matrix, may always be regarded as having an appropriately "ordered form" to begin with.

Since the sign matrices (59) and (60) for the tree configurations of Fig. 7 are too simple to represent worth-

[2] E. A. Guillemin, "How to grow your own trees from given cut-set or tie-set matrices," IRE TRANS. ON CIRCUIT THEORY, vol. CT-6, pp. 110–126; May, 1959.

while examples, and moreover, since they are based upon a particular branch numbering sequence that need not be fulfilled in an arbitrarily given situation, we choose to illustrate the method of tree construction proposed here by the following more elaborate sign matrix

$$S = \begin{bmatrix} + & + & + & + & + & + & + & + & + \\ & + & + & + & + & + & + & - & + \\ & & + & - & - & + & - & + & + \\ & & & + & + & - & + & + & + \\ & & & & + & - & - & + & + \\ & & & & & + & - & + & + \\ & & & & & & + & + & + \\ & & & & & & & + & - \\ & & & & & & & & + \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \qquad (61)$$

Numbering of the rows and columns is done to facilitate identification with correspondingly numbered tree branches.

Construction of the tree is accomplished by starting with the last branch, 9, and successively adding branches $8, 7, 6, \cdots$, being guided as to their relative positions by the confluence or counterfluence of reference arrows as demanded by the signs in the respective rows and columns of the matrix S. Thus branches 8 and 9 must be counterfluent while branch 7 is confluent with both 8 and 9. This state of affairs can be met only by having branches 7, 8, and 9 meet in a common point as shown in Fig. 8(b).
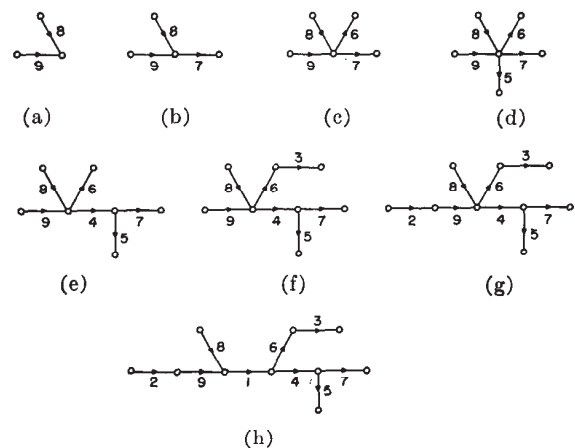

Fig. 8.

Fig. 8(a) to 8(h) shows the growth of the tree, branch by branch, the position of each added branch being uniquely determined (except for trivial variants) by the signs in the pertinent row of S. Construction of the tree is thus straightforward and always possible unless one encounters a contradiction, in which case no tree exists and the given G matrix has no realization.

Final Realization Procedure
for the Matrix G

Once the tree for a given G matrix has been found, it is a simple matter to write down the transformation matrix T which, in a congruent transformation, carries G over into a form appropriate either to a starlike tree (the dominant form) or to a linear tree (the uniformly tapered form). In either case the conditions for its realization present no further problem. Finally, the terminal pairs appropriate to the given matrix can readily be determined from the geometrical picture upon which construction of the matrix T is based.

When trivial variants in the tree structure exist, the following reasoning removes the pertinent ambiguities. For example, in Fig. 8(h) there are two trivial variants. Branches 3 and 6 can be interchanged and so can branches 2 and 9. If we leave them as they are, then in the G matrix appropriate to this tree, the familiar physical interpretation of what the elements in G stand for tells us that the element (13) must be less than the element (16); and the element (12) must be less than the element (19). These conditions can readily be checked; if fulfilled, the G matrix must be realizable with the tree as it stands if it is realizable at all. If either condition is not fulfilled, then we must consider the appropriate variant in the tree structure or else make the pertinent row-and-column interchange in G.

Observe that the existence of a tree appropriate to the given G matrix is not sufficient to assure its realization. Observe also that although we can construct many different T matrices connecting the tree of the given G matrix with a starlike or linear tree, it is sufficient to try only one, for if this one fails to yield a realizable G matrix (either dominant or uniformly tapered) then no other transformation T can do so, since a contrary assumption leads to a contradiction. Hence the realization procedure discussed here does not require repeated trials; one straightforward attack tells the whole story.

Conclusion

When the given G matrix contains zero elements[3] then the sign matrix correspondingly contains blank spaces which may be interpreted either as plus or minus signs. If, in the construction of the tree in the above described manner, either sign is admissable for the insertion of a particular tree branch, then a variant in the tree's tentative configuration is possible. If such a variant is non-trivial, and if non-trivial variants occur again in subsequent steps of the tree-growth process, then more than one geometrical tree pattern can be associated with the given G matrix. However, it need not follow that the realizability conditions appropriate to these various trees are fulfilled for the given element values in G.

Although several possibilities need now to be investigated, their number is greatly reduced from the totality of algebraic sign arrangements possible on a purely combinatorial basis by reason of the step-by-step nature of the tree-growing process which enables one by inspection to rule out the majority of trials and hence keep the procedure well within reasonable bounds even in the consideration of degenerate cases.

Collaterally it may be interesting to point out that the present discussion offers an alternate method for the construction of trees (and hence graphs) from given cut-set matrices. Forming the Grammian from the rows of α yields a G matrix appropriate to a network with all 1-ohm branches. From its sign matrix the pertinent tree can be constructed (if one exists); and situations for which other methods fail[4] do not necessarily lead to a degenerate G matrix and hence are strictly routine when handled by the method presented here.

At this point we need to remind ourselves that if a given G matrix is not realizable by an $n + 1$-node network it may still be realizable by a $2n$-node network in which the terminal pairs to which G pertains can be nonadjacent as they manifestly must be in the $n + 1$-node graph where each tree branch presents a terminal pair. In a $2n$-node full graph one can always choose a linear tree, and if we identify alternate branches as terminal pairs we have a topological situation that is completely general so far as the realizability of G is concerned. In other words a linear tree in a $2n$-node full graph with every other tree branch yielding an accessible terminal pair is the most general topological structure upon which the realization of G can be based. Additional nodes beyond the number $2n$ can do no good since any G realizable with more than $2n$ nodes is also realizable with $2n$ nodes because the familiar star-mesh transformation process eliminates the extra nodes and leaves all elements positive.

Before the realization techniques discussed here can be applied to this more general situation, however, one must augment the given $n \times n$ G matrix to one having $2n-1$ rows and columns. This augmentation process must, of course, fulfill the condition that subsequent abridgement to the accessible terminal-pairs regains the originally given $n \times n$ G matrix.

Such an augmentation process can be formulated in simple terms and, as might be expected, it is not unique. Herein lies the greater realization capability of the $2n$-node network over the $n + 1$-node network, for one can use the freedom inherent in the non-uniqueness of the augmentation process to help fulfill realization conditions for the implied $2n$-node graph with linear tree.

The details of this process, which presents some difficulties not yet fully resolved, will be presented here in the near future.

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[3] Omitting the consideration of situations in which the network consists of more than one separate part, the test for which is described in Guillemin, *ibid.*

[4] See in particular the one pertinent to the graph in Fig. 8, p. 126 of Guillemin, *ibid.*